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Collective String Field Theory of Matrix Models in the BMN Limit

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This work is dedicated to the memory of Bunji Sakita.

We develop a systematic procedure for deriving canonical string field theory from large N matrix models in the Berenstein-Maldacena-Nastase limit. The approach, based on collective field theory, provides a generalization of standard string field theory.

1. Introduction

Berenstein, Maldacena and Nastase (BMN) have established a correspondence between the spectrum of closed superstrings in the pp-wave background and $\mathcal{N} = 4$ super Yang-Mills theory [1]. In the process, they defined a (double scaling) limit of Yang-Mills theory with large N and large R-charge in which string theory is to be obtained. This limit simultaneously simplifies and extends the previous AdS/CFT correspondence in which the map was understood for supergravity states [2],[3],[4]. It represents a significant step in establishing the long held expectation that large N gauge theories at nonperturbative level lead to string theory [5],[6],[7],[8],[9],[10],[11],[12]. The new correspondence has the promise of providing a gauge theoretic description of string theory in a variety of backgrounds.

Vigorous studies are now being carried out in establishing [13],[14],[15],[16],[17] the Berenstein, Maldacena and Nastase correspondence. Much of the focus represents an effort to understand the interactions generated by Yang-Mills theory. Positive results but with certain limitations have been obtained in the literature [18],[19],[20],[21],[22],[23],[24],[25],[26],[27]. One of the difficulties is that short of direct comparison of amplitudes (between pp-wave strings and YM correlators) no systematic procedure was developed. The comparison in question has been limited at the present time to a set of correlators termed perturbative, but for a much larger set of so-called non-perturbative correlators, it still remains to be established. Related are issues of unitarity [28] for the light-cone type field theory. A proposal for a holographic map that was given in [29].

In this work we formulate a systematic scheme of obtaining canonical string field theory in the BMN limit. The method we apply is that of collective field theory [7],[30],[31], which provides a clear and well defined scheme for making the transition from YM to a string theory description. The method was employed successfully in establishing the first matrix model/string field theory map, namely that of non-critical strings [32], [33]. By its nature the method represents a direct change of variables from the matrices of $U(N)$ gauge theory to the fields of string theory. We focus in this paper on generic matrix models for notational reason as they are sufficient in demonstrating the basics of the present method. The simplest example of the derivation of string type interactions is given already in the free model. For SUGRA type amplitudes this essentially gives the full result. The effect of YM interactions and g_{YM} corrections will be given the next paper[34].

Our plan is as follows: After the Introduction in Sect.1, we summarize the basics of collective field theory in Sect.2. In Sect.3 we give a simple example with permutation

symmetry. In this case we discuss the passage from time-like gauge field theory to the light cone which is of relevance when performing a comparison with previous works. In Sect.4 we discuss the free matrix model in the BMN limit, concentrating on supergravity type modes. For these, Yang-Mills type interactions do not play a significant role and we exhibit agreement with SFT already at this level. In sect.5 we discuss general 'stringy states' exhibiting the corresponding 3-string interaction.

In Sect.6 under the conclusions, we discuss the generality of collective string field theory (CSFT) in comparison with standard light cone field theory.

2. Collective field theory

In this section, we give a general overview of collective field theory [30],[31]. It represents a systematic formalism for describing the dynamics of invariant observables of the theory. In gauge or matrix theory the physical observables are given by loops or traces of matrix products (words). The method consists of a direct change of variables to the invariant observables. The result is a (collective) Hamiltonian describing the full dynamics of the theory.

To be specific, let us consider a complex multi-matrix system with Hamiltonian

$$H = -\text{Tr}\left(\sum_{i=1}^M \frac{\partial}{\partial \bar{Z}_i} \frac{\partial}{\partial Z_i}\right) + V(\bar{Z}_i, Z_i). \quad (2.1)$$

where the potential $V(\bar{Z}_i, Z_i)$ is invariant under

$$Z_i \rightarrow U^\dagger Z_i U \quad \bar{Z}_i \rightarrow U^\dagger \bar{Z}_i U.$$

Then, the Hamiltonian is invariant under the above symmetry, and one may consider equal time single trace correlators (operators) of the form

$$\text{Tr}\left(\dots \prod_{i=1}^M Z_i^{n_i} \bar{Z}_i^{\bar{n}_i} \prod_{j=1}^M Z_j^{m_j} \bar{Z}_j^{\bar{m}_j} \dots\right). \quad (2.2)$$

In the large N limit, the change of variables from the original variables to loop variables implies a reduction of degrees of freedom. For instance, in atomic systems or single matrix systems the collective (loop) variables correspond to the radial component and eigenvalue basis, respectively. There is by now ample evidence coming both from studies of single matrix models [35] and matrix descriptions of lower dimensional strings [32],

[33],[36],[37],[38] that these variables can be treated as independent in the large N limit. Possible constraints due to the finite size of matrices can always be imposed after the change of variables. This results in interesting effects related to the large N exclusion principle [39].

Let us denote the invariant field variables by

$$\phi_C$$

where C is a loop or word index. For simplicity of notation, we require this index to include $\bar{\phi}_C$. One may then consider changing variables from the original variables to this new set [30]. Concentrating on the kinetic term, one has:

$$T = -\text{Tr}\left(\sum_{i=1}^M \frac{\partial}{\partial \bar{Z}_i} \frac{\partial}{\partial Z_i}\right) = -\sum_{C,C'} \Omega(C, C') \frac{\partial}{\partial \phi_C} \frac{\partial}{\partial \phi_{C'}} + \sum_C \omega(C) \frac{\partial}{\partial \phi_C} \quad (2.3)$$

where

$$\Omega(C, C') = \text{Tr}\left(\sum_{i=1}^M \frac{\partial \bar{\phi}_C}{\partial \bar{Z}_i} \frac{\partial \phi_{C'}}{\partial Z_i}\right) = \bar{\Omega}(C', C)$$

and

$$\omega(C) = -\text{Tr}\left(\sum_{i=1}^M \frac{\partial^2 \phi_C}{\partial \bar{Z}_i \partial Z_i}\right)$$

$\Omega(C, C')$ “joins” loops, or words. As an example, if $\phi_C = \text{Tr}(Z_1^J)$ and $\phi_{C'} = \text{Tr}(Z_1^{J'})$ then $\Omega = JJ' \text{Tr}(Z_1^{J-1} \bar{Z}_1^{J'-1})$. So in general, one may write schematically

$$\Omega(C, C') = \sum \phi_{C+C'}$$

where $C + C'$ is obtained by adding the two words C and C' . Similarly, ω “splits” loops. Schematically again,

$$\omega(C) = \sum \phi_{C'} \phi_{C''} \quad (2.4)$$

represents all the processes of splitting the word C into C' and C'' .

As any change of variables the present one also involves a nontrivial Jacobian contributing to the measure. The Jacobian plays a central role in guaranteeing the unitarity

of the new description. A most relevant aspect in the construction of collective field theory is the determination of the Jacobian. By performing a similarity transformation

$$\frac{\partial}{\partial \phi_C} \rightarrow J^{1/2} \frac{\partial}{\partial \phi_C} J^{-1/2} = \frac{\partial}{\partial \phi_C} - \frac{1}{2} \frac{\partial \ln J}{\partial \phi_C}$$

one imposes the requirement that the Hamiltonian (2.3) becomes explicitly hermitean. This requirement of hermiticity provides a system of equations for the Jacobian. This set of equations establishes the necessary data for a complete rewriting of the hamiltonian in terms of the new observables. The differential equation is

$$-(\bar{\partial}_{C'} \ln J) \tilde{\Omega}(C', C) = \omega(C) + \bar{\partial}_{C'} \tilde{\Omega}(C', C) \quad (2.5)$$

with

$$\tilde{\Omega}(C, C') = \frac{1}{2} \left[\text{Tr} \left(\sum_{i=1}^M \frac{\partial \bar{\phi}_C}{\partial \bar{Z}_i} \frac{\partial \phi_{C'}}{\partial Z_i} \right) + \text{Tr} \left(\sum_{i=1}^M \frac{\partial \phi_{C'}}{\partial \bar{Z}_i} \frac{\partial \bar{\phi}_C}{\partial Z_i} \right) \right] = \tilde{\Omega}(\bar{C}', \bar{C})$$

The explicitly hermitean hamiltonian (collective field hamiltonian) is then seen to read

$$H = \left(\frac{\partial}{\partial \bar{\phi}(C)} + \frac{1}{2} \frac{\partial \ln J}{\partial \bar{\phi}_C} \right) \tilde{\Omega}(C, C') \left(-\frac{\partial}{\partial \phi(C')} + \frac{1}{2} \frac{\partial \ln J}{\partial \phi_{C'}} \right) + V$$

Once use is made of (2.5), the leading contribution to the collective field hamiltonian is then

$$H' = \left(-\frac{\partial}{\partial \bar{\phi}_C} \tilde{\Omega}(C, C') \frac{\partial}{\partial \phi_{C'}} + \frac{1}{4} \omega(C) \tilde{\Omega}^{-1}(C, C') \bar{\omega}(C') \right) + V \quad (2.6)$$

or

$$H' = \left(\bar{\Pi}(C) \tilde{\Omega}(C, C') \Pi(C') + \frac{1}{4} \omega(C) \tilde{\Omega}^{-1}(C, C') \bar{\omega}(C') \right) + V$$

where

$$\Pi(C) = -i \frac{\partial}{\partial \phi(C)} \quad \bar{\Pi}(C) = -i \frac{\partial}{\partial \bar{\phi}(C)}$$

The full Hamiltonian in addition contains counterterms which contribute at loop level. Their form is explicitly determined by the hermiticity requirement and reads

$$\begin{aligned} \Delta H = & -\frac{1}{2} \frac{\partial \omega(C)}{\partial \phi_C} + \frac{1}{2} \frac{\partial \tilde{\Omega}(C'', C')}{\partial \phi_{C''}} \tilde{\Omega}^{-1}(C', C) \bar{\omega}(C) \\ & + \frac{1}{4} \frac{\partial \tilde{\Omega}(C'', C)}{\partial \phi_{C''}} \tilde{\Omega}^{-1}(C, C') \frac{\partial \tilde{\Omega}(C', C''')}{\partial \phi_{C'''}} \end{aligned}$$

Let us now comment on some relevant features of the collective Hamiltonian. First of all any matrix model interaction will appear in this approach as a tadpole term (ie linear in the fields). As such they will have a role in the full theory (specific backgrounds correspond to minima of the effective potential $1/4 \omega \tilde{\Omega}^{-1} \bar{\omega} + V$) but the basic non linearity is seen to emerge from the free, kinetic term. These will be 'renormalized' by the matrix interactions. One also sees that in addition to simple cubic interactions, the formalism generates a sequence of higher point interactions(vertices). These come from the expansion of $1/4 \omega \tilde{\Omega}^{-1} \bar{\omega}$, the non trivial effective potential of collective field theory (for a recent study of this term see [40]). In addition one has the loop counterterms. All these are specified by the requirement of Hermiticity. The Hamiltonian (2.6) is sufficient to describe fluctuations about a given background [41]. Furthermore, as it will be described in the following, the term $\Pi \Omega \Pi$ will contain enough information to study the structure of three point interactions.

The non triviality in applying collective field theory to multimatrix models or Yang-Mills theory is contained in the following. In the context of interest to us in this paper, BMN have identified a set of observables (traces) which have a mapping into the pp wave string. If in the collective approach one begins with a given set of loops C and their conjugates C' , through the process of "joining" (contained in $\Omega(C, C')$) one generates new loops not in the original set. It is this sequence of extra degrees of freedom that we have to understand in the present approach. Their relevance will also point to a more general scheme contained in collective string field theory(CSFT).

3. Example With S_N Symmetry

In this section a simple system of particles with S_N symmetry will be given. This example is to illustrate the strategy we employ in our study of matrix models, for which the corresponding collective field theories are not yet explicitly available. Apart from illustrating the workings of collective field theory in this example we establish some facts about the large J limit which will be of general value. Start with the free N-body Hamiltonian

$$H = -2 \sum_{i=1}^N \frac{\partial}{\partial z_i} \frac{\partial}{\partial \bar{z}_i} + \frac{\omega^2}{2} \sum_{i=1}^N \bar{z}_i z_i. \quad (3.1)$$

where

$$z_i = x_i + iy_i, \quad \bar{z}_i = x_i - iy_i,$$

The relevant collective field variables are the S_N invariants

$$\phi_{nm} = \sum_{i=1}^N \bar{z}_i^n z_i^m. \quad (3.2)$$

After a Fourier transform, they become the density fields $\phi(\bar{Z}, Z)$. The change of variable results in the

$$\begin{aligned} H &= \int dxdy \left(2\partial_{\bar{z}}\Pi\phi\partial_z\Pi + \frac{1}{2} \frac{\partial_{\bar{z}}\phi\partial_z\phi}{\phi} + \frac{1}{2}\omega^2 z\bar{z}\phi \right) \\ &\equiv \int dxdy 2\partial_{\bar{z}}\Pi\phi\partial_z\Pi + \mathcal{V}[\phi]. \end{aligned} \quad (3.3)$$

The expansion of this Hamiltonian about the leading large N configuration will generate an infinite number of interaction vertices, despite the fact that the original system is noninteracting. The leading large N configuration

$$\phi_0 = \frac{N\omega}{\pi} e^{-\omega z\bar{z}}, \quad (3.4)$$

minimizes the effective potential $\mathcal{V}[\phi]$. Expanding about the leading configuration

$$\Pi = \frac{1}{\sqrt{\phi_0}} \tilde{\Pi}, \quad \phi = \phi_0 + \sqrt{\phi_0} \eta, \quad (3.5)$$

we obtain the following quadratic Hamiltonian

$$H_2 = \frac{1}{2} \int dxdy \left(\tilde{\Pi}(-4\partial_z\partial_{\bar{z}} + \omega^2 z\bar{z} - 2\omega)\tilde{\Pi} + \eta(-4\partial_z\partial_{\bar{z}} + \omega^2 z\bar{z} - 2\omega)\eta \right). \quad (3.6)$$

We now expand $\tilde{\Pi}$ and η

$$\eta = \sum_{J=1}^{\infty} \left(\bar{c}_J \psi_J + c_J \bar{\psi}_J \right), \quad \tilde{\Pi} = \sum_{J=1}^{\infty} \left(p_J \psi_J + \bar{p}_J \bar{\psi}_J \right) \quad (3.7)$$

in terms of the Harmonic oscillator eigenfunctions

$$\psi_J = \sqrt{\frac{\omega^{J+1}}{\pi J!}} z^J e^{-\frac{\omega}{2} z\bar{z}}, \quad \bar{\psi}_J = \sqrt{\frac{\omega^{J+1}}{\pi J!}} \bar{z}^J e^{-\frac{\omega}{2} z\bar{z}}, \quad (3.8)$$

which satisfy

$$\begin{aligned}
(-4\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}} + \omega^2 z \bar{z})\psi_J &= 2(J+1)\omega\psi_J, \\
(-4\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}} + \omega^2 z \bar{z})\bar{\psi}_J &= 2(J+1)\omega\bar{\psi}_J.
\end{aligned} \tag{3.9}$$

The quadratic Hamiltonian becomes

$$H_2 = \frac{1}{2} \int dx dy \left(J \omega p_J \bar{p}_J + J \omega c_J \bar{c}_J \right). \tag{3.10}$$

Notice that H_2 gives the exact spectrum of our toy model.

We now consider the interactions generated by the collective field theory. There are two contributions to the cubic vertex. First we have the momentum field dependent term

$$\int dx dy 2 \partial_z \Pi \sqrt{\phi_0} \eta \partial_{\bar{z}} \Pi$$

giving

$$\begin{aligned}
\sum_{J_1, J_2, J_3} 4\pi \mathcal{M} \int dr \ r^{J_1+J_2+J_3-1} e^{-\omega r^2} (\bar{c}_{J_2} \bar{p}_{J_3} p_{J_1} + c_{J_2} \bar{p}_{J_1} p_{J_3}) \delta_{J_2+J_1, J_3}. \\
\mathcal{M}^2 = \frac{N \omega^{J_1+J_2+J_3+2}}{\pi^2 N^2 J_1! J_2! J_3!}.
\end{aligned}$$

After integration over r we obtain

$$V_3 = \sum_{J_1, J_2, J_3} \omega \left(\frac{J_1 + J_2 + J_3}{2} - 1 \right)! \frac{2 J_1 J_3}{\sqrt{N J_1! J_2! J_3!}} (\bar{p}_{J_3} \bar{c}_{J_2} p_{J_1} + \bar{p}_{J_1} p_{J_3} c_{J_2}) \delta_{J_1+J_2, J_3}.$$

In an identical way one evaluates the three point interaction coming from the $\omega \Omega^{-1} \omega = \frac{\partial_{\bar{z}} \phi \partial_z \phi}{\phi}$ term. When expanded in terms of the fluctuating field this term generates an infinite series of vertices. At the cubic level it gives:

$$\tilde{V}_3 = - \sum_{J_1, J_2, J_3} \frac{\omega}{16} \left(\frac{J_1 + J_2 + J_3}{2} - 1 \right)! \frac{(J_1 + J_2 + J_3)^2}{\sqrt{N J_1! J_2! J_3!}} (\bar{c}_{J_1} \bar{c}_{J_2} c_{J_3} + \bar{c}_{J_3} c_{J_1} c_{J_2}) \delta_{J_1+J_2, J_3}.$$

At this point we pause and discuss the result and make some relevant comments. The collective hamiltonian that we have constructed is a hamiltonian in a time-like gauge. The (collective) fields c_J and \bar{c}_J represent left and right movers (with J being a momentum). These modes correspond to observables with a pure z^J or a \bar{z}^J dependence. In addition

we have the fields p_J and \bar{p}_J which correspond to an insertion of a time derivative of Z . Studies of string theory in the pp-wave background discuss construction of a light cone hamiltonian (it was in fact notoriously difficult to construct interacting string field theory hamiltonians in a time like gauge). From our time-like hamiltonian we can pass to a light cone one by taking the infinite momentum (large J) limit. Introduce creation-annihilation operators as usual

$$\begin{aligned} c_J &= a_J + \tilde{a}_J^\dagger, & \bar{c}_J &= a_J^\dagger + \tilde{a}_J, \\ p_J &= -\frac{i}{2}(a_J^\dagger - \tilde{a}_J), & \bar{p}_J &= \frac{i}{2}(a_J - \tilde{a}_J^\dagger) \end{aligned} \quad (3.11)$$

Passage to the lightcone frame is now implemented by dropping the \tilde{a} , \tilde{a}^\dagger oscillators. This represents a reduction of degrees of freedom. Reducing the $\Pi\Omega\Pi = \partial_{\bar{z}}\Pi\phi\partial_z\Pi$ term contribution to just right movers gives

$$\sum_{J_1, J_2, J_3} \frac{\omega}{4} \sqrt{\frac{J_3!}{N J_1! J_2!}} (J_1 + J_2) (a_{J_1}^\dagger a_{J_2}^\dagger a_{J_3} + a_{J_3}^\dagger a_{J_1} a_{J_2}) \delta_{J_1+J_2, J_3}.$$

Our second interaction term gives

$$- \sum_{J_1, J_2, J_3} \frac{\omega}{4} J_3 \sqrt{\frac{J_3!}{N J_1! J_2!}} (a_{J_1}^\dagger a_{J_2}^\dagger a_{J_3} + a_{J_3}^\dagger a_{J_1} a_{J_2}) \delta_{J_1+J_2, J_3}.$$

Summing these two contributions we obtain the light cone cubic vertex

$$H_3 = \sum_{J_1, J_2, J_3} \frac{1}{4} \sqrt{\frac{J_3!}{N J_1! J_2!}} (J_1\omega + J_2\omega - J_3\omega) (a_{J_1}^\dagger a_{J_2}^\dagger a_{J_3} + a_{J_3}^\dagger a_{J_1} a_{J_2}) \delta_{J_1+J_2, J_3}. \quad (3.12)$$

Here we notice that after summing the resulting vertex comes with a prefactor equal to the difference in energies of the participating modes. Thus this vertex is seen to vanish on shell. We also see that the prefactor responsible for this is the one proposed in [20]. We emphasize that our full timelike gauge interaction does not vanish (on shell), it is only after the projection to light cone frame that we obtain the vanishing prefactor. This cancellation came from the two types of terms and represents a general property of collective field interaction when projected to the infinite momentum frame.

It is not obvious how this vertex could possibly reproduce the three point functions of the original model. To explore this point, note that equal time correlation functions of z_i and \bar{z}_i map into the equal time correlation functions of modes of the collective field

$$\begin{aligned}
& \langle \left(\int d^2 x_1 z_1^{J_1} \phi(x_1) \right) \left(\int d^2 x_2 z_2^{J_2} \phi(x_2) \right) \left(\int d^2 x_3 \bar{z}_3^{(J_1+J_2)} \phi(x_3) \right) \rangle \\
&= \langle \sum_{i=1}^N (x_i + iy_i)^{J_1} \sum_{j=1}^N (x_j + iy_j)^{J_2} \sum_{k=1}^N (x_k - iy_k)^{J_1+J_2} \rangle.
\end{aligned} \tag{3.13}$$

The correlators in the original model are easily evaluated

$$\langle \sum_{i=1}^N (x_i + iy_i)^{J_1} \sum_{j=1}^N (x_j + iy_j)^{J_2} \sum_{k=1}^N (x_k - iy_k)^{J_1+J_2} \rangle = N \frac{(J_1 + J_2)!}{\omega^{J_1+J_2}}. \tag{3.14}$$

Now consider the collective field theory calculation. Performing the integrations

$$\begin{aligned}
& \langle \left(\int d^2 x_1 z_1^{J_1} \phi(x_1, t) \right) \left(\int d^2 x_2 z_2^{J_2} \phi(x_2, t) \right) \left(\int d^2 x_3 \bar{z}_3^{(J_1+J_2)} \phi(x_3, t) \right) \rangle \\
&= \sqrt{\frac{N^3 J_1! J_2! J_3!}{\omega^{J_1+J_2+J_3}}} \langle A_{J_1} A_{J_2} A_{J_3}^\dagger(t) \rangle.
\end{aligned}$$

where we have the dressed creation-annihilation operators. In terms of first order perturbation theory these receive a contribution given by the cubic interactions

$$\frac{1}{4} \sqrt{\frac{J_3!}{N J_1! J_2!}} (\omega J_1 + \omega J_2 - \omega J_3)$$

multiplied by the (propagators) factor

$$\frac{2}{\omega J_1 + \omega J_2 - \omega J_3},$$

which neatly cancels the energy prefactor contained in the cubic vertex reproducing the correlator (3.14).

In this simple toy model, we see that the structure of the light cone cubic vertex is seen by the $\Pi\Omega\Pi$ term of the collective field theory Hamiltonian. The net effect of the $\omega\Omega^{-1}\omega$ term is to complete the cubic vertex so that it comes multiplied by the correct prefactor. Thus, to compute the vertex up to the prefactor we would only need to consider the $\Pi\Omega\Pi$ term. This appears to be a rather general conclusion. The collective field theory also provides a natural explanation of the difference in energies prefactor multiplying the cubic vertex.

4. Collective String Field Theory of Matrix Models

The BMN study addressed the $\mathcal{N} = 4$ Super Yang-Mills theory in four dimensions. In this particular case arguments were presented for a correspondence with string theory in the ppwave background. Through investigation of correlation functions of the theory it became visible that some of the basic structure is carried by the matrix (Higgs) fields of the theory. It is these degrees of freedom that we will concern ourselves with in our study. Clearly the full (Yang-Mills) theory is relevant for the actual correspondence and our discussion of the present matrix model (done for notational simplicity) should be viewed in that context. The degrees of freedom that we follow consist then of the Higgs fields: $\Phi_1 \Phi_2 \cdots \Phi_5, \Phi_6$. It will be sufficient to consider them as functions of time only which corresponds to matrix QM. This represents a sector of the full theory that we study presently. Two of the Higgs matrices (Φ_5, Φ_6) are chosen to play a special role in the BMN scheme, defining the light cone momenta $Z = \Phi_5 + i\Phi_6$. We denote the other complex Higgs as Y_i $i = 1, 2$. In the spirit of collective field theory, the structure of interactions will come from considering the free $d = 1$ Hamiltonian

$$H = \text{Tr} \left(-\frac{\partial}{\partial \bar{Z}} \frac{\partial}{\partial Z} + \mu^2 \bar{Z} Z \right) + \text{Tr} \left(-\frac{\partial}{\partial \bar{Y}_i} \frac{\partial}{\partial Y_i} + \mu^2 \bar{Y}_i Y_i \right), \quad (4.1)$$

since we are interested only in string coupling constant ($= \frac{1}{N}$) interactions.

4.1. Supergravity Amplitudes

Our basic loop variables are

$$\begin{aligned} O^J &= \text{Tr}(Z^J), & \bar{O}^J &= \text{Tr}(\bar{Z}^J), \\ \Pi^J &= \frac{\partial}{\partial O^J}, & \bar{\Pi}^J &= \frac{\partial}{\partial \bar{O}^J}. \end{aligned} \quad (4.2)$$

The two point function of our loops is

$$\langle O^J \bar{O}^J \rangle = J N^J \left(\frac{1}{2\mu} \right)^J.$$

We rescale our collective field variables

$$\Pi'^J = \sqrt{J \left(\frac{N}{2\mu} \right)^J} \Pi^J, \quad O^J = \sqrt{J \left(\frac{N}{2\mu} \right)^J} O'^J,$$

so that they have a normalized two point function. Our strategy is to consider the contribution to the collective field theory Hamiltonian arising from the $(\Pi\Omega\Pi)$ loop joining term. Experience from the toy model suggests this is enough to reproduce the cubic vertex, up to the prefactor. The loop joining contribution is

$$\begin{aligned} T &= \sum_{J_1, J_2} J_1 J_2 \text{Tr}(Z^{J_1-1} \bar{Z}^{J_2-1}) \Pi^{J_1} \bar{\Pi}^{J_2} \\ &\equiv \sum_{J_1, J_2} \Omega(J_1, J_2) \Pi^{J_1} \bar{\Pi}^{J_2}. \end{aligned} \quad (4.3)$$

It is not surprising that the joining process has produced a loop which is not among the variables we consider. To proceed further, we need to express $\Omega(J_1, J_2)$ in terms of the loops we consider.

We could imagine expanding $\Omega(J_1, J_2)$, as an operator, about its large N value. The leading term in this expansion would give the contribution to the quadratic Hamiltonian. It is a simple task to evaluate the expectation value

$$\langle \text{Tr}(Z^{J_1-1} \bar{Z}^{J_2-1}) \rangle = \delta_{J_1 J_2} N^{J_1} \left(\frac{1}{2\mu} \right)^{J_1-1} \left(1 + O(N^{-2}) \right). \quad (4.4)$$

Using the leading term and expressing T in terms of the rescaled variables, we obtain

$$T = \sum_J 2J_\mu \Pi'^J \bar{\Pi}'^J.$$

This is consistent with the exact spectrum of chiral loops, if we assume no mixing at the quadratic level.

We have seen that collective field theory generates a new set of loops. These contain both Z s and \bar{Z} s and their expectation (or classical) value was already needed for quadratic fluctuations. More completely we factorize these loops into a product containing our basic (holomorphic) loops and the ones with nonzero classical expectation values. The factorization is achieved through the loop splitting formula (2.4). Indeed one can write

$$\frac{\partial}{\partial Z_{ij}} (Z^{J_1-1} \bar{Z}^{J_2-2})_{ij} = \frac{\partial}{\partial Z_{ij}} \left((Z^{J_1-1} \bar{Z}^{J_2-2})_{lk} \frac{\partial \bar{Z}_{kl}}{\partial \bar{Z}_{ji}} \right) = \sum_{J_3=0}^{J_1-2} \text{Tr}(Z^{J_1-2-J_3} \bar{Z}^{J_2-2}) \text{Tr}(Z^{J_3})$$

and the splitting is obtained as the derivative with respect to Z_{ij} acts through the loop. This splitting rule is also the main ingredient in the Schwinger-Dyson equations for matrix

models. It is known [9] that these equations follow from the equations of motion of this effective collective field theory action. This leads us to the factorization equation

$$\text{Tr}(Z^{J_1-1}(t)\bar{Z}^{J_2-1}(t)) \Rightarrow \frac{A_{J_1 J_2}}{2\mu} \sum_{J_3=0}^{J_1-2} \text{Tr}(Z^{J_1-2-J_3}(t)\bar{Z}^{J_2-2}(t))O^{J_3}(t), \quad (4.5)$$

for the composite loops. We also have

$$\text{Tr}(Z^{J_2-1}(t)\bar{Z}^{J_1-1}(t)) \Rightarrow \frac{B_{J_1 J_2}}{2\mu} \sum_{J_3=0}^{J_1-2} \text{Tr}(\bar{Z}^{J_1-2-J_3}(t)Z^{J_2-2}(t))\bar{O}^{J_3}(t), \quad (4.6)$$

which can be derived in a similar way. The constant factors $A_{J_1 J_2}$, $B_{J_1 J_2}$ in above formulas represent normalization and can be fixed by looking at the expectation values of both sides of this relation. The method of taking expectations of both sides can itself be used to fully specify the index structure of the form factor appearing in the above formula(see [34]). Note that the large N expectation values of O^J is zero and that of $\text{Tr}(Z^{J_1-2-J_3}(t)\bar{Z}^{J_2-2}(t))$ is given in (4.4). Expanding each operator in (4.5) and (4.6) about their leading large N value, then to linear order in the fluctuations we obtain

$$\begin{aligned} \text{Tr}(Z^{J_1-1}\bar{Z}^{J_2-1}) &= N^{J_1} \left(\frac{1}{2\mu}\right)^{J_1-1} \delta_{J_1 J_2} + J_2 \sum_{J_3=1}^{J_1-2} \delta_{J_2+J_3, J_1} N^{J_2-1} \left(\frac{1}{2\mu}\right)^{J_2-1} O_{J_3} \\ &+ J_1 \sum_{J_3=1}^{J_2-2} \delta_{J_1+J_3, J_2} N^{J_1-1} \left(\frac{1}{2\mu}\right)^{J_1-1} \bar{O}_{J_3}. \end{aligned} \quad (4.7)$$

We will take this as an ansatz for the structure of $\Omega(J_1, J_2)$ to linear order in the fluctuations. Using this ansatz, we find that the loop joining contribution to the cubic vertex is

$$2\mu \frac{\sqrt{J_1 J_2 J_3}}{N} (J_1 \bar{O}'^{J_3} \Pi'^{J_1} \bar{\Pi}'^{J_2} \delta_{J_3+J_1, J_2} + J_2 O^{J_3} \Pi'^{J_1} \bar{\Pi}'^{J_2} \delta_{J_3+J_2, J_1}). \quad (4.8)$$

As we have discussed in the example of sect.3 the transition to light cone fields is achieved through replacement of canonical (conjugate) fields by creation (annihilation) fields. In addition we expect analogous contribution from the collective potential so that the final form of the cubic interaction becomes:

$$\begin{aligned}
H_3 = & \frac{\sqrt{J_1 J_2 J_3}}{N} \delta_{J_1, J_2 + J_3} \left(\frac{J_2}{J_1} \right)^{\frac{n_2}{2}} \left(\frac{J_3}{J_1} \right)^{\frac{n_3}{2}} (J_1 - J_2 - J_3) A_{J_1}^\dagger A_{J_2} A_{J_3} \\
& + \frac{\sqrt{J_1 J_2 J_3}}{N} \delta_{J_1, J_2 + J_3} \left(\frac{J_2}{J_1} \right)^{\frac{n_2}{2}} \left(\frac{J_3}{J_1} \right)^{\frac{n_3}{2}} (J_1 - J_2 - J_3) A_{J_2}^\dagger A_{J_3}^\dagger A_{J_1}.
\end{aligned}$$

We would now like to consider loops with one type of impurity added. We identify ϕ with Y_1 in (4.1) and consider the following loop variables

$$\begin{aligned}
O_n^J &= \sum \text{Tr}(\phi^n Z^J), & \bar{O}_n^J &= \sum \text{Tr}(\bar{\phi}^n \bar{Z}^J), \\
\Pi_n^J &= \frac{\partial}{\partial O_n^J}, & \bar{\Pi}_n^J &= \frac{\partial}{\partial \bar{O}_n^J}.
\end{aligned} \tag{4.9}$$

The sums in the definition of the loops run over all possible orderings of n ϕ s and J Z s, i.e. there are $\frac{(n+J)!}{n!J!}$ terms in the above sums. We will work in the BMN limit[1], so that $J \gg n$. In this limit, the two point functions of our loops are

$$\begin{aligned}
\langle O_n^{J_1} \bar{O}_m^{J_2} \rangle &= \delta_{J_1 J_2} \delta_{mn} \left(\frac{1}{2\mu} \right)^{J_1 + n} N^{J_1 + n} (J_1 + n) \frac{(J_1 + n)!}{J_1! n!} \\
&= \delta_{J_1 J_2} \delta_{mn} N^{J_1 + n} \frac{J_1^{n+1}}{n!} \left(\frac{1}{2\mu} \right)^{J_1 + n}.
\end{aligned} \tag{4.10}$$

We will again work in terms of the normalized loop variables

$$\Pi_n'^J = \sqrt{\left(\frac{N}{2\mu} \right)^{J+n} \frac{J^{n+1}}{n!}} \Pi_n^J, \quad O_n^J = \sqrt{\left(\frac{N}{2\mu} \right)^{J+n} \frac{J^{n+1}}{n!}} O_n'^J.$$

The loop joining contribution to the Hamiltonian is given by

$$H = \text{Tr} \left(\frac{\partial O_{n_1}^{J_1}}{\partial Z} \frac{\partial \bar{O}_{n_2}^{J_2}}{\partial \bar{Z}} + \frac{\partial O_{n_1}^{J_1}}{\partial \phi} \frac{\partial \bar{O}_{n_2}^{J_2}}{\partial \bar{\phi}} \right) \Pi_{n_1}^{J_1} \bar{\Pi}_{n_2}^{J_2}. \tag{4.11}$$

For the manipulations which follow, it is convenient to introduce the matrix $P(J, n)$, defined to be the matrix obtained by summing the $\frac{(n+J)!}{J!n!}$ terms corresponding to all possible orderings of J Z s and n ϕ s. In terms of $P(J, n)$ we have

$$\frac{\partial O_n^J}{\partial Z_{ij}} = (J + n) P(J - 1, n)_{ji} \quad \frac{\partial O_n^J}{\partial \phi_{ij}} = (J + n) P(J, n - 1)_{ji}. \tag{4.12}$$

We can express (4.11) in terms of the $P(J, n)$ matrices as

$$\begin{aligned}
H &= (J_1 + n_1)(J_2 + n_2) \text{Tr} \left(P(J_1 - 1, n_1) \bar{P}(J_2 - 1, n_2) \right. \\
&\quad \left. + P(J_1, n_1 - 1) \bar{P}(J_2, n_2 - 1) \right) \Pi_{n_1}^{J_1} \bar{\Pi}_{n_2}^{J_2} \\
&\equiv \Omega(J_1, n_1, J_2, n_2) \Pi_{n_1}^{J_1} \bar{\Pi}_{n_2}^{J_2}.
\end{aligned} \tag{4.13}$$

In the large N limit we have

$$\langle \text{Tr} (P(J_1, n_1) \bar{P}(J_2, n_2)) \rangle = \delta_{J_1 J_2} \delta_{n_1 n_2} N^{J_1 + n_1 + 1} \frac{1}{(2\mu)^{J_1 + n_1}} \frac{(n_1 + J_1)!}{n_1! J_1!}. \tag{4.14}$$

Proceeding as we did for the loops with no impurities, we obtain the following contribution to the quadratic Hamiltonian

$$T = \sum_{J, n} 2(J + n) \mu \Pi_{n_1}^{\prime J_1} \bar{\Pi}_{n_2}^{\prime J_2}. \tag{4.15}$$

This again matches the exact spectrum for a loop with one type of impurity provided we again assume that there is no mixing between these modes at quadratic level. To obtain the cubic vertex, we need the analog of (4.5), (4.6) to motivate an ansatz for $\Omega(J_1, n_1, J_2, n_2)$. To obtain the relevant Schwinger-Dyson equation, we will need to use the identities

$$\begin{aligned}
P(J, m) &= P(J - 1, m)Z + P(J, m - 1)\phi \\
\frac{\partial P(J, m)_{ij}}{\partial Z_{kl}} &= \sum_{r=0}^m \sum_{S=0}^{J-1} P(S, r)_{ik} P(J - S - 1, m - r)_{lj} \\
\frac{\partial P(J, m)_{ij}}{\partial \phi_{kl}} &= \sum_{r=0}^{m-1} \sum_{S=0}^J P(S, r)_{ik} P(J - S, m - r - 1)_{lj}.
\end{aligned} \tag{4.16}$$

Using these identities, we are lead to the following splitting formula for the composite operators

$$\begin{aligned}
&\text{Tr}(P(J_1 - 1, n_1) [\bar{P}(J_2 - 2, n_2) \bar{Z} + \bar{P}(J_2 - 1, n_2 - 1) \bar{\phi}]) \\
&+ \text{Tr}(P(J_1, n_1 - 1) [\bar{P}(J_2 - 1, n_2 - 1) \bar{Z} + \bar{P}(J_2, n_2 - 2) \bar{\phi}]) \\
&\Rightarrow \frac{1}{2\mu} \sum_{r=0}^{n_1} \sum_{S=0}^{J_1-2} \text{Tr}(P(S, r)) \text{Tr}(P(J_1 - S - 2, n_1 - r) \bar{P}(J_2 - 2, n_2)) \\
&+ \frac{1}{\mu} \sum_{r=0}^{n_1-1} \sum_{S=0}^{J_1-1} \text{Tr}(P(S, r)) \text{Tr}(P(J_1 - S - 1, n_1 - r - 1) \bar{P}(J_2 - 1, n_2 - 1)) \\
&+ \frac{1}{2\mu} \sum_{r=0}^{n_1-2} \sum_{S=0}^{J_1} \text{Tr}(P(S, r)) \text{Tr}(P(J_1 - S, n_1 - r - 2) \bar{P}(J_2, n_2 - 2)).
\end{aligned} \tag{4.17}$$

Noting that

$$O_n^J = \text{Tr}(P(J, n),$$

the identity (4.17) suggests the following ansatz for $\Omega(J_1, n_1, J_2, n_2)$

$$\begin{aligned} \Omega(J_1, n_1, J_2, n_2) = & \delta_{J_1, J_2} \delta_{n_1, n_2} (J_1 + n_1)(J_2 + n_2) \frac{(J_1 + n_1)!}{J_1! n_1!} \frac{N^{J_1 + n_1 + 1}}{(2\mu)^{J_1 + n_1}} \\ & + \delta_{J_1, J_2 - J_3} \delta_{n_1, n_2 - n_3} (J_1 + n_1)^2 (J_2 + n_2) \frac{(J_1 + n_1)!}{J_1! n_1!} \left(\frac{N}{2\mu}\right)^{J_2 + n_2} \bar{O}_{n_3}^{J_3} \\ & + \delta_{J_1 - J_3, J_2} \delta_{n_1 - n_3, n_2} (J_1 + n_1)(J_2 + n_2)^2 \frac{(J_2 + n_2)!}{J_2! n_2!} \left(\frac{N}{2\mu}\right)^{J_1 + n_1} O_{n_3}^{J_3}. \end{aligned}$$

Using this ansatz, the loop joining contribution to the cubic vertex becomes

$$\begin{aligned} & \frac{\sqrt{J_1 J_2 J_3}}{N} 2\mu \sqrt{\frac{n_1!}{n_2! n_3!}} \left(\frac{J_2}{J_1}\right)^{\frac{n_2}{2}} \left(\frac{J_3}{J_1}\right)^{\frac{n_3}{2}} (J_2 + n_2) \delta_{J_1, J_2 + J_3} \delta_{n_1, n_2 + n_3} O_{n_3}^{J_3} \Pi_{n_1}'^{J_1} \bar{\Pi}_{n_2}'^{J_2} \\ & + \frac{\sqrt{J_1 J_2 J_3}}{N} 2\mu \sqrt{\frac{n_2!}{n_1! n_3!}} \left(\frac{J_1}{J_2}\right)^{\frac{n_1}{2}} \left(\frac{J_3}{J_2}\right)^{\frac{n_3}{2}} (J_1 + n_1) \delta_{J_2, J_1 + J_3} \delta_{n_2, n_1 + n_3} \bar{O}_{n_3}^{J_3} \Pi_{n_1}'^{J_1} \bar{\Pi}_{n_2}'^{J_2}. \end{aligned} \quad (4.18)$$

As discussed previously, the transition to light cone fields is achieved through replacement of canonical (conjugate) fields by creation (annihilation) fields. Again adding the analogous contribution from the collective potential the final form of the cubic interaction becomes

$$\begin{aligned} H_3 = & \frac{\sqrt{J_1 J_2 J_3}}{N} \sqrt{\frac{n_1!}{n_2! n_3!}} \delta_{J_1, J_2 + J_3} \delta_{n_1, n_2 + n_3} \left(\frac{J_2}{J_1}\right)^{\frac{n_2}{2}} \left(\frac{J_3}{J_1}\right)^{\frac{n_3}{2}} \\ & \times ((J_1 + n_1) - (J_2 + n_2) - (J_3 + n_3)) (A_{J_1, n_1}^\dagger A_{J_2, n_2} A_{J_3, n_3} + A_{J_2, n_2}^\dagger A_{J_3, n_3}^\dagger A_{J_1, n_1}). \end{aligned}$$

in agreement with the cubic interaction matrix elements for bosonic supergravity modes, as given for example in [24].

The above argument can be extended to loops that include more than just one impurity. In order to illustrate this point, we will give the generalization to two impurities. Identifying ϕ and ψ with Y_1 and Y_2 of (4.1) respectively, we consider the following loop variables

$$O_{n, m}^J = \sum \text{Tr}(\phi^n \psi^m Z^J). \quad (4.19)$$

The sum on right hand side again runs over all possible orderings of the matrix fields, i.e. there are $\frac{(n+m+J)!}{n!m!J!}$ terms in the sum. The two point functions of our loops are

$$\begin{aligned} \langle O_{n_1, m_1}^{J_1} \bar{O}_{n_2, m_2}^{J_2} \rangle &= \delta_{J_1 J_2} \delta_{m_1 m_2} \delta_{n_1 n_2} \left(\frac{N}{2\mu} \right)^{J_1 + n_1 + m_1} \\ &\quad (J_1 + n_1 + m_1) \frac{(J_1 + n_1 + m_1)!}{J_1! n_1! m_1!}. \end{aligned} \quad (4.20)$$

The operators with unit two point function are obtained by rescaling

$$\begin{aligned} \Pi_{n_1, m_1}'^{J_1} &= \sqrt{\left(\frac{N}{2\mu} \right)^{J_1 + n_1 + m_1} (J_1 + n_1 + m_1) \frac{(J_1 + n_1 + m_1)!}{J_1! n_1! m_1!}} \Pi_{n_1, m_1}^{J_1}, \\ O_{n_1, m_1}^{J_1} &= \sqrt{\left(\frac{N}{2\mu} \right)^{J_1 + n_1 + m_1} (J_1 + n_1 + m_1) \frac{(J_1 + n_1 + m_1)!}{J_1! n_1! m_1!}} O_{n_1, m_1}'^{J_1}. \end{aligned}$$

It is again convenient to introduce the matrix $M(J, n, m)$ which is a sum over the $\frac{(J+n+m)!}{J!m!n!}$ terms obtained by constructing all possible arrangements of J Z fields, m ψ fields and n ϕ fields. In terms of $M(J, n, m)$ we have

$$\frac{\partial O_{n, m}^J}{\partial Z_{ij}} = (J + m + n) M(J - 1, n, m)_{ji},$$

$$\frac{\partial O_{n, m}^J}{\partial \phi_{ij}} = (J + m + n) M(J, n - 1, m)_{ji},$$

$$\frac{\partial O_{n, m}^J}{\partial \psi_{ij}} = (J + m + n) M(J, n, m - 1)_{ji},$$

As before, we consider only the loop joining contribution to the collective field theory Hamiltonian

$$\begin{aligned} H &= \Omega(J_1, n_1, m_1, J_2, n_2, m_2) \Pi_{n_1, m_1}^{J_1} \bar{\Pi}_{n_2, m_2}^{J_2} \\ &= \left(\frac{\partial O_{n_1, m_1}^{J_1}}{\partial Z_{ij}} \frac{\partial \bar{O}_{n_2, m_2}^{J_2}}{\partial \bar{Z}_{ji}} + \frac{\partial O_{n_1, m_1}^{J_1}}{\partial \phi_{ij}} \frac{\partial \bar{O}_{n_2, m_2}^{J_2}}{\partial \bar{\phi}_{ji}} + \frac{\partial O_{n_1, m_1}^{J_1}}{\partial \psi_{ij}} \frac{\partial \bar{O}_{n_2, m_2}^{J_2}}{\partial \bar{\psi}_{ji}} \right) \Pi_{n_1, m_1}^{J_1} \bar{\Pi}_{n_2, m_2}^{J_2}. \end{aligned}$$

Using the two point function

$$\begin{aligned} \langle \text{Tr}(M(J_1, n_1, m_1) \bar{M}(J_2, n_2, m_2)) \rangle &= \delta_{J_1 J_2} \delta_{m_1 m_2} \delta_{n_1 n_2} \\ &\quad \times \frac{(J_1 + n_1 + m_1)!}{J_1! n_1! m_1!} \frac{N^{J_1 + n_1 + m_1 + 1}}{(2\mu)^{J_1 + n_1 + m_1}} \end{aligned}$$

we find the following contribution to the quadratic Hamiltonian

$$H = (J + n + m)\Pi'_{n,m}{}^J \bar{\Pi}'_{n,m}{}^J.$$

To obtain the ansatz for $\Omega(J_1, n_1, m_1, J_2, n_2, m_2)$ needed to obtain the cubic vertex, we use the identities

$$M(J, n, m) = M(J - 1, n, m)Z + M(J, n - 1, m)\phi + M(J, n, m - 1)\psi,$$

$$\frac{\partial M(J, n, m)_{ij}}{\partial Z_{kl}} = \sum_{r=0}^{J-1} \sum_{s=0}^n \sum_{t=0}^m M(r, s, t)_{ik} M(J - 1 - r, n - s, m - t)_{lj},$$

$$\frac{\partial M(J, n, m)_{ij}}{\partial \phi_{kl}} = \sum_{r=0}^J \sum_{s=0}^{n-1} \sum_{t=0}^m M(r, s, t)_{ik} M(J - r, n - s - 1, m - t)_{lj},$$

$$\frac{\partial M(J, n, m)_{ij}}{\partial \psi_{kl}} = \sum_{r=0}^J \sum_{s=0}^n \sum_{t=0}^{m-1} M(r, s, t)_{ik} M(J - r, n - s, m - t - 1)_{lj},$$

We obtain the following identity between $d = 1$ operators

$$\begin{aligned}
& \text{Tr} \left(M(J_1 - 1, n_1, m_1) \bar{M}(J_2 - 1, n_2, m_2) + M(J_1, n_1 - 1, m_1) \bar{M}(J_2, n_2 - 1, m_2) + \right. \\
& \left. + M(J_1, n_1, m_1 - 1) \bar{M}(J_2, n_2, m_2 - 1) \right) \\
& \Rightarrow \frac{1}{2\mu} \sum_{J=0}^{J_1-2} \sum_{n=0}^{n_1} \sum_{m=0}^{m_1} \text{Tr}(M(J, n, m)) \text{Tr} \left(M(J_1 - 2 - J, n_1 - n, m_1 - m) \bar{M}(J_2 - 2, n_2, m_2) \right) \\
& + \frac{1}{\mu} \sum_{J=0}^{J_1-1} \sum_{n=0}^{n_1-1} \sum_{m=0}^{m_1} \text{Tr}(M(J, n, m)) \text{Tr} \left(M(J_1 - 1 - J, n_1 - n - 1, m_1 - m) \right. \\
& \quad \left. \times \bar{M}(J_2 - 1, n_2 - 1, m_2) \right) \\
& + \frac{1}{\mu} \sum_{J=0}^{J_1-1} \sum_{n=0}^{n_1} \sum_{m=0}^{m_1-1} \text{Tr}(M(J, n, m)) \text{Tr} \left(M(J_1 - 1 - J, n_1 - n, m_1 - m - 1) \right. \\
& \quad \left. \times \bar{M}(J_2 - 1, n_2, m_2 - 1) \right) \\
& + \frac{1}{2\mu} \sum_{J=0}^{J_1} \sum_{n=0}^{n_1-2} \sum_{m=0}^{m_1} \text{Tr}(M(J, n, m)) \text{Tr} \left(M(J_1 - J, n_1 - n - 2, m_1 - m) \bar{M}(J_2, n_2 - 2, m_2) \right) \\
& + \frac{1}{\mu} \sum_{J=0}^{J_1} \sum_{n=0}^{n_1-1} \sum_{m=0}^{m_1-1} \text{Tr}(M(J, n, m)) \text{Tr} \left(M(J_1 - J, n_1 - n - 1, m_1 - m - 1) \right. \\
& \quad \left. \times \bar{M}(J_2, n_2 - 1, m_2 - 1) \right) \\
& + \frac{1}{2\mu} \sum_{J=0}^{J_1} \sum_{n=0}^{n_1} \sum_{m=0}^{m_1-2} \text{Tr}(M(J, n, m)) \text{Tr} \left(M(J_1 - J, n_1 - n, m_1 - m - 2) \bar{M}(J_2, n_2, m_2 - 2) \right).
\end{aligned}$$

Using this equation to motivate an ansatz in our usual fashion, we land up with the following loop joining contribution to the cubic vertex

$$\begin{aligned}
& 2\mu \delta_{J_1, J_2+J_3} \delta_{n_1, n_2+n_3} \delta_{m_1, m_2+m_3} \frac{\sqrt{J_1 J_2 J_3}}{N} \sqrt{\frac{n_1!}{n_2! n_3!}} \sqrt{\frac{m_1!}{m_2! m_3!}} \\
& \times (J_2 + n_2 + m_2) \left(\frac{J_2}{J_1} \right)^{\frac{n_2+m_2}{2}} \left(\frac{J_3}{J_1} \right)^{\frac{n_3+m_3}{2}} \Pi'^{J_1}_{n_1, m_1} \bar{\Pi}'^{J_2}_{n_2, m_2} O'^{J_3}_{n_3, m_3}.
\end{aligned} \tag{4.21}$$

This again implies a light cone coupling of the form

$$(\Delta J + \Delta n + \Delta m) \sqrt{\frac{n_1!}{n_2! n_3!}} \sqrt{\frac{m_1!}{m_2! m_3!}} \delta_{J_1, J_2+J_3} \delta_{n_1, n_2+n_3} \delta_{m_1, m_2+m_3} \left(\frac{J_2}{J_1} \right)^{\frac{n_2+m_2}{2}} \left(\frac{J_3}{J_1} \right)^{\frac{n_3+m_3}{2}}$$

with $\Delta J = J_1 + J_2 - J_3$ and similarly for n and m . This is in agreement with the cubic interaction matrix elements for bosonic supergravity modes.

5. Lattice string

In this section we show that in position (lattice) space the collective field theory produces the ultralocal vertex of string field theory.

Our general collective field variables are now given by

$$\Phi_J(\{l\}) = \text{Tr} \left(T_l \prod_{i=1}^n \phi(l_i) Z^J \right), \quad \phi(l_i) = Z^{l_i} \phi Z^{-l_i}, \quad (5.1)$$

with T_l the l ordering operator - it orders the $\phi(l)$ factors so that l_i increases from left to right. As an example, the correspondence between collective fields and states in the ultralocal string field theory Hilbert space is given by

$$\Phi_J(\{l_i, l_i, l_i, l_j, l_j\}) \leftrightarrow \frac{(a^\dagger(i))^3}{\sqrt{3!}} \frac{(a^\dagger(j))^2}{\sqrt{2!}} |0, J\rangle. \quad (5.2)$$

We also have

$$\bar{\Phi}_J(\{l\}) = \text{Tr} \left(\bar{Z}^J \tilde{T}_l \prod_{i=1}^n \bar{\phi}(l_i) \right), \quad \bar{\phi}(l_i) = \bar{Z}^{-l_i} \bar{\phi} \bar{Z}^{l_i}. \quad (5.3)$$

\tilde{T}_l is a second l ordering operator - it orders the $\bar{\phi}(l)$ factors so that l_i decreases from left to right. The loops are orthogonal at large N , i.e.

$$\langle \Phi_J(\{k\}) \bar{\Phi}_{\bar{J}}(\{l\}) \rangle = \delta_{J\bar{J}} \prod_{i=1}^n \delta_{k_i l_i} \frac{1}{(2\mu)^{J+n}}.$$

Introduce

$$P_J(\{l\})_{ij} = \sum_{a=1}^J T_l \left(\left[\prod_{i=1}^n \phi(l_i - a) Z^{J-1} \right]_{ij} \right) = \frac{\partial \Phi_J(\{l\})}{\partial Z_{ji}}$$

In the last formula we take $l_i - a \bmod J$ so that $0 \leq l_i - a \leq J-1$. Introduce

$$Q_J(\{l\})_{ij} = \sum_{j=1}^n T_l \left(\prod_{i=1, i \neq j}^n \left[\phi(l_i - l_j) Z^J \right]_{ij} \right) = \frac{\partial \Phi_J(\{l\})}{\partial \phi_{ji}}$$

In the last formula we take $l_i - l_j \bmod J$. At leading order in N we have

$$\langle \text{Tr} \left[T_l \left(\prod_{i=1}^n \phi(l_i) \right) Z^J \bar{Z}^{\bar{J}} \tilde{T}_l \left(\prod_{j=1}^{\bar{n}} \bar{\phi}(\bar{l}_j) \right) \right] \rangle = \delta_{n\bar{n}} \delta_{J\bar{J}} N^{n+J+1} \prod_{i=1}^n \delta_{l_i \bar{l}_i} \frac{1}{(2\mu)^{J+n}}.$$

Next, we need to study the quantity

$$\begin{aligned}
& \frac{\partial \Phi_{J_1}(\{l\})}{\partial Z_{ij}} \frac{\partial \bar{\Phi}_{J_2}(\{\bar{l}\})}{\partial \bar{Z}_{ji}} + \frac{\partial \Phi_{J_1}(\{l\})}{\partial \phi_{ij}} \frac{\partial \bar{\Phi}_{J_2}(\{\bar{l}\})}{\partial \bar{\phi}_{ji}} \\
&= \text{Tr}(P_{J_1}(\{l\})\bar{P}_{J_2}(\{\bar{l}\})) + \text{Tr}(Q_{J_1}(\{l\})\bar{Q}_{J_2}(\{\bar{l}\})) \\
&= \sum_{a=1}^{J_1} \sum_{b=1}^{J_2} \text{Tr} \left[T_l \left(\prod_{i=1}^{n_1} \phi(l_i - a) \right) Z^{J_1-1} \bar{Z}^{J_2-1} \tilde{T}_l \left(\prod_{j=1}^{n_2} \bar{\phi}(\bar{l}_j - b) \right) \right] \\
&+ \sum_{k=1}^{n_1} \sum_{m=1}^{n_2} \text{Tr} \left[T_l \left(\prod_{i=1, i \neq k}^{n_1} \phi(l_i - l_k) \right) Z^{J_1} \bar{Z}^{J_2} \tilde{T}_l \left(\prod_{j=1, j \neq m}^{n_2} \bar{\phi}(\bar{l}_j - \bar{l}_m) \right) \right]
\end{aligned}$$

It is helpful to introduce

$$\begin{aligned}
P'_J(\{l\})_{ij} &= \sum_{\substack{a=1, \\ l_i - a \neq J-1}}^J T_l \left(\prod_{i=1}^n \left[\phi(l_i - a) Z^{J-2} \right]_{ij} \right) \\
\tilde{P}_J(\{l\})_{ij} &= \sum_{\substack{j=1, \\ i \neq j}}^n T_l \left(\prod_{i=1}^n \left[\phi(l_i - l_j - 1) Z^{J-1} \right]_{ij} \right)
\end{aligned}$$

To get P'_J from P_J , keep all terms that end with a Z and strip off the last Z . To get \tilde{P}_J from P_J , keep all terms ending with a ϕ and strip off the last ϕ .

$$P'_J(\{l\})Z + \tilde{P}_J(\{l\})\phi = P_J(\{l\})$$

It is straight forward to verify that

$$\begin{aligned}
\frac{\partial P_{J_1}(\{l\})_{kl}}{\partial Z_{ij}} &= \sum_{a=1}^{J_1} \sum_{J_3=0}^{J_1-1} \left(T_l \left[\prod_{\substack{i=1 \\ l_i - a \leq J_3 \\ l_i - a \neq J_1 - 1}}^n \phi(l_i - a) Z^{J_3} \right]_{ki} \right) \\
&\times \left(T_l \left[\prod_{\substack{i=1 \\ l_i - a \geq J_3 + 1 \\ l_i - a \neq J_1 - 1}}^n \phi(l_i - a - J_3 - 1) Z^{J_1 - J_3 - 2} \right]_{jl} \right)
\end{aligned}$$

The analysis of the supergravity modes given in the previous section can be used to argue that we can drop $\text{Tr}(\frac{\partial}{\partial\phi}\frac{\partial}{\partial\phi})$ and keep just the $\text{Tr}(\frac{\partial}{\partial\bar{Z}}\frac{\partial}{\partial\bar{Z}})$ term. For this reason, concentrate on

$$\begin{aligned} & \frac{\partial\Phi_{J_1}(\{l\})}{\partial Z_{ij}} \frac{\partial\bar{\Phi}_{J_2}(\{\bar{l}\})}{\partial\bar{Z}_{ji}} \\ &= \sum_{a=1}^{J_1} \sum_{b=1}^{J_2} \text{Tr} \left[T_l \left(\prod_{i=1}^{n_1} \phi(l_i - a) \right) Z^{J_1-1} \bar{Z}^{J_2-1} \tilde{T}_l \left(\prod_{j=1}^{n_2} \bar{\phi}(\bar{l}_j - b) \right) \right], \end{aligned}$$

writting it as

$$\begin{aligned} \text{Tr}(P_{J_1}(\{l\})\bar{P}_{J_2}(\{\bar{l}\})) &= \frac{\partial}{\partial Z_{ij}} \left([P_{J_1}(\{l\})\bar{P}'_{J_2}(\{\bar{l}\})]_{ij} \right) \\ &+ \frac{\partial}{\partial\phi_{ij}} \left([P_{J_1}(\{l\})\bar{\bar{P}}_{J_2}(\{\bar{l}\})]_{ij} \right). \end{aligned}$$

Keeping only the leading term we obtain

$$\begin{aligned}
\text{Tr}(P_{J_1}(\{l\})\bar{P}_{J_2}(\{\bar{l}\})) &\Rightarrow \frac{1}{2\mu} \sum_{a=1}^{J_1} \sum_{J_3=0}^{J_1-1} \text{Tr} \left(T_l \left[\prod_{i=1}^{n_3} \phi(l_i - a) Z^{J_3} \right] \right. \\
&\quad \left. \begin{array}{l} l_i - a \leq J_3 \\ l_i - a \neq J_1 - 1 \end{array} \right) \\
&\times \text{Tr} \left(T_l \left[\prod_{i=1}^{n_1-n_3} \phi(l_i - a - J_3 - 1) \right] Z^{J_1-J_3-2} \bar{P}'_{J_2}(\{\bar{l}\}) \right) \\
&\quad \begin{array}{l} l_i - a \geq J_3 + 1 \\ l_i - a \neq J_1 - 1 \end{array} \\
&= \frac{1}{2\mu} \sum_{a=1}^{J_1} \sum_{J_3=0}^{J_1-1} \text{Tr} \left(T_l \left[\prod_{i=1}^{n_3} \phi(p_i) Z^{J_3} \right] \right) \times \\
&\quad \prod_{i=1}^{n_3} \delta_{p_i, l_i - a} \prod_{i=1}^{n_1-n_3} \delta_{q_i, l_i - a - J_3 - 1} \\
&\quad \begin{array}{l} l_i - a \leq J_3 \\ l_i - a \neq J_1 - 1 \end{array} \quad \begin{array}{l} l_i - a \geq J_3 + 1 \\ l_i - a \neq J_1 - 1 \end{array} \\
&\times \text{Tr} \left(T_l \left[\prod_{i=1}^{n_1-n_3} \phi(q_i) \right] Z^{J_1-J_3-2} \bar{Z}^{J_2-2} \tilde{T}_l \left[\prod_{i=1}^{n_2} \bar{\phi}(\bar{l}_i) \right] \right) \\
&= \frac{1}{2\mu} \sum_{a=1}^{J_1} \sum_{J_3=0}^{J_1-1} \delta_{\{p\}, \{l\}} \delta_{\{q\}, \{\bar{l}\}} \\
&\Phi_{J_3}(\{p\}) \text{Tr} \left(T_l \left[\prod_{i=1}^{n_3-n_2} \phi(q_i) \right] Z^{J_1-J_3-2} \bar{Z}^{J_2-2} \tilde{T}_l \left[\prod_{i=1}^{n_2} \bar{\phi}(\bar{l}_i) \right] \right)
\end{aligned}$$

After rescaling and normalization, our cubic interaction takes the form

$$\frac{2\mu}{N} (J_1 + n_1) \delta_{J_1-J_3, J_2} \delta_{n_1-n_3, n_2} \prod_{i=1}^{n_2} \delta_{q_i, \bar{l}_i} \prod_{j=1}^{n_3} \delta_{p_j, q_j+n_2} \Phi_{J_3}(\{p\}) \Pi_{J_1}(\{q\}) \bar{\Pi}_{J_2}(\{\bar{l}\})$$

Again in the light cone limit and with the matching contribution coming from the collective potential, we are lead to the following 3-string interaction:

$$\begin{aligned}
H_3^{col} &= \\
&\sum_{J_1, J_2, J_3} \sum_{\{l^{(1)}\}, \{l^{(2)}\}, \{l^{(3)}\}} (\Delta J + \Delta n) \langle \psi_1 | \langle \psi_2 | \langle \psi_3 | V_3^0 \Phi_{J_3}(\{l^{(3)}\}) \frac{\partial}{\partial \Phi_{J_1}(\{l^{(1)}\})} \frac{\partial}{\partial \Phi_{J_2}(\{l^{(2)}\})} \\
&+ h.c.
\end{aligned}$$

Here the prefactor again contains the energies in the form $(E_3^0 - E_1^0 - E_2^0)$ and $E^0 = J + n$. This interaction vertex of collective field theory matches up with the string field theory vertex. It is represented by

$$|V\rangle = e^{\sum_{l=0}^{\alpha_1-1} a_1^\dagger(l) a_3^\dagger(l) + \sum_{l=\alpha_1}^{\alpha_3-1} a_2^\dagger(l-\alpha_1) a_3^\dagger(l)} |0\rangle.$$

Using the notation (we order the l_i 's so that $l_i \geq l_j$ if $i > j$)

$$|\alpha_1, \{l_1, n_1\}, \{l_2, n_2\}, \dots, \{l_N, n_N\}\rangle \equiv \frac{(a(l_1)^\dagger)^{n_1}}{n_1!} \frac{(a(l_2)^\dagger)^{n_2}}{n_2!} \dots \frac{(a(l_N)^\dagger)^{n_N}}{n_N!} |0, \alpha_1\rangle,$$

for a state with occupation numbers n_i at sites l_i , we find

$$\begin{aligned} & \langle V | \alpha_1, \{l_1, n_1\}, \dots, \{l_N, n_N\} \rangle_1 | \alpha_2, \{m_1, p_1\}, \dots, \{m_M, p_M\} \rangle_2 \\ & | \alpha_3, \{q_1, r_1\}, \dots, \{q_{N+M}, r_{N+M}\} \rangle_3 \\ & = \delta_{\alpha_1+\alpha_2, \alpha_3} \delta_{l_1 q_1} \delta_{n_1 r_1} \dots \delta_{l_N q_N} \delta_{n_N r_N} \delta_{m_1+\alpha_1, q_{N+1}} \delta_{p_1 r_{N+1}} \\ & \quad \times \dots \delta_{m_M+\alpha_1, q_{N+M}} \delta_{p_M r_{N+M}}. \end{aligned} \tag{5.4}$$

demonstrating the agreement.

6. Conclusions

We have shown in the present work how based on the collective field method one generates an interacting string field theory from the dynamics of matrices. The interactions are seen to originate from the free (kinetic) term of a matrix theory, they come proportional to $1/N$. In general, gauge theory interactions will renormalize these basic string interactions. The described method offers a potential for constructing both types (perturbative and nonperturbative) of interactions. In the earlier application of the theory to noncritical strings one has not differentiated between the two. We will present this discussion in a future publication.

The method being based on direct, time-like hamiltonian formalism is fundamentally unitary. That is the function of the (highly) nontrivial Jacobian transformation that defines collective field theory. Apart from leading to a sequence of higher point vertices it also provides necessary counterterms whose effect becomes relevant at the loop level (see [33] for examples of such calculations). The present scheme then offers a potential for direct and complete map from matrix theory to string field theory.

Collective string field theory (CSFT) amplitudes as a rule agree with results computed by other means. But it is relevant to stress that CSFT represents a generalization and a broader framework compared to standard string field theory (SFT). While standard light-cone SFT is set in a fixed background (for example the pp wave) in collective string field theory (CSFT) the background is determined as a solution of the (collective) equation. Equivalently it can be specified by the evaluation of 'one-point' functions (Wilson loops) in Yang-Mills theory. The full theory exhibits a number of degrees of freedom much larger than that contained in standard SFT. As we have demonstrated the extra fields play a relevant role in establishing the background since their expectation values are seen to be nonzero. In this very fundamental sense CSFT represents an extension of customary SFT.

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